

THE MATHEMATICS TEACHER

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VOLUME VI

JUNE, 1914

NUMBER 4

Dr. Wheelock's paper gives some very interesting statistics concerning the success of different schools in the examinations in mathematics set by the New

A Danger.

York State Regents Department.

There is some danger however in giving too much weight to these facts and the conclusions derived from them. Every experienced teacher knows that it is possible to obtain a very high percentage of success in such examinations as these by definitely preparing the pupils for the examination, rather than working for their interests and individual needs.

It is often said that the second purpose, if the work is well done, will include the former. The fact remains, nevertheless, that the number of possible combinations of geometric principles used in an examination is necessarily limited, and a teacher who centralizes the course about previous examination questions and makes the passing of the examination the aim of teaching, may obtain a high degree of success in this, without giving the pupil any adequate training. On the other hand, the teacher who wishes to broaden the outlook of his pupils and to give to each lasting interest in such methods of thought as geometry furnishes will many times feel hampered by the necessity of reaching even the best chosen examination standard, which by its very existence, fits the norm, not the varying units.

These facts do not go to prove that ability to pass an examination is not a desirable thing, but simply that our testing systems are still far from perfect, and that the teacher should test the results of his own teaching far more strictly than can any system.

ARE PARTICULAR ABILITIES NECESSARY FOR
PUPILS TO GAIN AN UNDERSTANDING OF
THE ELEMENTARY AND SECONDARY
MATHEMATICS AS USUALLY GIVEN
AT THE PRESENT TIME.

BY CHARLES F. WHEELOCK.

The student of mathematics in the elementary school and the high school is called upon to deal with three quite different and distinctly marked kinds of subject matter.

The first deals principally with the technical language of mathematics. This technical language is made up largely of pure conventions. There is no reasoning required, either inductive or deductive. All that is required is a knowledge of simply facts; for example, a mathematician uses a vertical cross to indicate multiplication. There is nothing in this sign itself that makes it indicate multiplication any more than division. It indicates multiplication simply because all those who use it have agreed that this shall be its meaning. Again, the expression $5a^3$ is written as it is written simply because people have agreed to write it so. If it had been agreed to write the third power of a taken five times as $3a^5$, we would have to learn it that way. The use and the meaning of the ordinary signs and symbols of mathematics come in the first instance under the head of things to be simply learned. Time was when the normal child could learn such things and the results reached in some schools today indicate that in some places it is being done today. It is, however, a lamentable fact that a large proportion of failures in mathematical study results from half learned mathematical speech. No particular intellectual faculty is required for the successful pursuit of this division of the work. There is nothing in it that requires any particular bulging of any particular part of the cranium. In fact, a healthy student might almost leave his cerebrum at home if his cerebellum were in good working order and still if he would get right down to hard work he might thoroughly master this phase of the subject of elementary and secondary mathematics. To the same

kind of subject matter belongs facility in performing ordinary operations in arithmetic and algebra and an adequate knowledge of a considerable class of algebraic expressions that are continually coming under observation. Just as we would expect the child to know the factors of 12 or 27 or 45 without thinking, without hesitation, so we would expect him to know the factors of $a^2 - b^2$, and of a large number of other type forms, and these are all of that class of matter that primarily require no reasoning. The child must know at sight the necessary fundamental things and must be able to do automatically the fundamental elementary operations. There is no reason why every child of ordinary ability can not reach a high degree of perfection in this particular field.

The second class of questions has to do with things that may be apprehended by the senses or may be represented to the imagination. Most of the subject matter in the so-called practical arithmetic and in the geometry, plane and solid, is made up of this sort of matter. A student may see a bushel of potatoes, a cord of wood, a gallon of oil, a pound of sugar, and having once seen, he may be called on to think of larger or smaller quantities of these substances and may be expected to build up a mental picture which he can hold up before himself while he is determining relations or comparing proportions. Having once seen and examined critically a board and a post, it should be possible to construct in his mind a post and board fence conforming to any reasonable conditions. To do things of this sort requires only that the pupil shall have senses capable of perceiving, a memory capable of holding, and a power of visualization capable of bringing back a picture when required and of changing it at will. A serious defect in any one of these qualities of mind will make satisfactory results in elementary arithmetic and geometry impossible, but the normal child has all of these faculties, and, moreover, has infinite capacity of development. If a teacher is sufficiently wise to inquire just wherein the pupil is weak and shape instruction and training toward strengthening the weak spot, I can see no good reason why the normal child should not succeed fairly well in this division of the work.

The third kind of subject matter is less tangible, less concrete, includes the abstractions of mathematics and requires a develop-

ment of the reasoning faculties. Howiston in the introduction of one of the chapters of his analytical geometry makes essentially this statement: "A general locus is something which exists to abstract thought, but it can be neither drawn nor imagined." This third class of questions deals with things that can not be drawn or imagined, but which are still very real. The advanced arithmetic and the advanced algebra includes about all there is of this kind of material in the high school course. To successfully grapple with questions relating to this matter, the student must be thoroughly grounded in the knowledge and the art mentioned in the first class of questions, and, in addition, have a trained memory that will enable him to hold in mind the premises of his problem and the results of successive steps from the beginning to the end of the course of reasoning, and he must have a power of concentrating his attention that will enable him to carry on his course through several successive steps. Students will differ widely in their ability to successfully handle matters of this sort, just as they will differ widely in their ability to lift weights, to run a race, or to do any other thing, physical or intellectual, but I can see no reason why the normal child should not be able to make some progress in this direction.

So far I have given opinions only. I wish now to state some facts to sustain these opinions. As you all know, we have in this state a system of state-conducted examinations covering the high school field. You may or you may not be believers in examinations, but I think you will agree that if an examination is a fair test anywhere, it is in mathematics. I give in tabular form the results in all the high school mathematics in eight different schools in this state, covering a period of five years, and including a total of 19,249 papers written in ten different examinations. I have designated the schools by the letters, *A, B, C, D*, etc.:

	Examined.	Accepted.	Per Cent.
<i>A</i>	5,590	5,484	about 98
<i>B</i>	1,081	811	about 75
<i>C</i>	2,407	1,813	about 80
<i>D</i>	924	729	about 79
<i>E</i>	2,063	1,643	about 80
<i>F</i>	2,749	2,075	about 76
<i>G</i>	1,448	1,390	about 95
<i>H</i>	2,984	2,544	about 85

It will be noted that in school "A" there were 5,590 papers written; 5,484 were accepted. In school "G" 1,448 papers were written and 1,390 accepted. Now it must be admitted that if for a period of five years, a record of this sort can be maintained, no exceptional ability on the part of the pupil is required for success, unless we are to conclude that for some unquestioned and unquestionable reason, all the high school pupils in these two communities have for a period of five years possessed "these particular abilities." In all of the eight schools whose results are here tabulated, a fairly large proportion of all the students have given evidence of making a fairly successful record in mathematics. The result becomes still more conclusive when we remember that algebra and plane geometry are subjects required of all the students in the schools mentioned.

I am giving next a table showing the results in eight other schools, from which a different conclusion might have been reached if we had not the record given in the first table.

	Examined.	Accepted.	Per Cent.
<i>I</i>	3,819	2,015	about 53
<i>J</i>	618	409	about 65
<i>K</i>	795	493	about 62
<i>L</i>	917	551	about 60
<i>M</i>	631	349	about 55
<i>N</i>	1,797	781	about 43
<i>O</i>	318	130	about 41
<i>P</i>	920	614	about 67

It is very evident that the marked differences in the results in the two groups of schools that I have mentioned are not due to differences in the mental capacity of the children attending these schools. The difference may be due to any one of many causes or to a combination of many causes. It may be due to defective school organization, to the lack of a proper educational spirit in the school and in the community, to a lack of knowledge on the part of both teacher and student as to what constitutes a proper standard of excellence, and hence satisfaction with inferior results, all of which are only different forms of the results of poor teaching, for good teaching should lead to sound educational ideals not only among the pupils, but in the community at large, and should thereby establish reasonable educational standards.

Every teacher of wide experience has known pupils who thoroughly believed that it was impossible for them to make satisfactory progress in certain subjects. One student is very sure that he can not learn Latin. He has no aptitude for the study of foreign languages. Another is equally certain that while he makes good progress in Latin, he is utterly unable to deal satisfactorily with mathematics. A very large proportion of all the students are very certain that they can not do much with music and drawing and English composition. In my opinion, students who hold these ideas are in general mistaken, and the mistake has been due to poor teaching at some stage of their progress. I have known many pupils who were such failures in mathematics that they had become thoroughly convinced that it was impossible for them to make any progress in mathematical study, who, on a change of teachers, have been transformed into excellent students in mathematics. The same holds true of every other department of knowledge. Again, and from the same cause, a whole school may become obsessed by the notion that certain departments in the school curriculum are extremely difficult and other departments are so easy as to be classified under the well-known term "snap courses,"—courses in which credits may be easily obtained. The interesting fact about this is that while in school "A" mathematics may be considered as very difficult and Latin as very easy, in school "B" everybody passes in geometry, but only the most brilliant succeed in getting through the Latin. I have in mind a school where there was a wide range of election, in which every student desired to take geometry because every student in that school always passed geometry, while in the majority of schools of the same character, geometry was dreaded because of the difficulty in getting a passing mark. Suggestion plays a large part in the mental attitude of a pupil in school toward the subject matter in his course of study. Tradition also is a potent influence. Let it once become generally felt throughout a school that geometry is a very difficult subject to master and geometry becomes a thing to be dreaded, to be approached with fear and trembling, and very likely to be repeated before a passing mark is reached. In many colleges such a relief is felt when a class has once finished with trigonometry, analytical geometry or calculus, that the event is celebrated and

the old textbooks are burned in the celebration, but in one college it will be trigonometry, in another it will be analytical geometry, in another it will be calculus that has been handed down by tradition as the bugbear.

It is, of course, true that there are occasional individual examples of students whose mental development is abnormal to such an extent that the best teaching and the best environment fail to produce results in some subjects, but I believe there is not sufficient evidence to lead us to the conclusion that such cases are more common than cases of physical deformity.

It will not do for us teachers of mathematics to try to account for the inferior results of mathematical study in so many schools by laying it to the inferior mental capacity of the children. "What man has done, man can do." If in one group of schools, the students are uniformly successful in mathematical study and if in another group of schools under similar environment, the results of such instruction are unsatisfactory, there can be only one conclusion; that is, that in one case the teaching is better than in the other case. What children have done and are continually doing in one school, similar children can do in another school similarly situated, if they have equally good instruction. We will strengthen the weak places only when we are willing to recognize that "it is in ourselves that we are underlings."

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SOME EXPERIENCES WITH A CLASS OF REVIEWERS IN GEOMETRY.

BY SARA C. WALSH.

There was offered, at the Buffalo Central High School last January, a review geometry class in an attempt to aid, in a particular way, those who had failed in the regents' examination one or more times. On looking up the records of the applicants to the course, it was discovered that some had taken it two or more times and in one extreme case, the boy had registered ten terms. None of these pupils were mentally deficient, as was shown by exceptional work in other lines. Such remarks as these reached me at the first regular session of the class: "I'll *never* get it." "What's the good of it anyhow?" "I hate the stuff." "I'd like to get together all the geometry books in the school and watch them burn."

I made a few remarks somewhat as follows: "At my own request, this class was given to me because I am convinced that *anyone* can pass geometry who can do as creditable work in other lines as some of you can. There are in this class, to my knowledge, two excellent musicians and two with oratorical ability and these four people claim that they cannot do geometry. Perhaps it is great success in one line that discourages us at mediocre success in another. Should we give up because things are difficult? I think the reasons for a large share of failure in geometry are (1) that pupils do not try; (2) that a misconception creeps in or something is swallowed wholesale because the book says so or the teacher says so. You must work with me or I shall become discouraged and that would be the worst thing that could happen to you. I know that some of you are discouraged but this is an opportunity freely to discuss anything you want, without the feeling that some superior fellow student will glance condescendingly at you. You are all in the same boat. Will it move onward or sink?"

I racked my brain in a desire to make geometry resemble something entirely unfamiliar. We assumed a knowledge of

the definitions of angles, triangles, quadrilaterals, axioms with dozens of examples from common experience and the conditions under which triangles are congruent. So far, resignation was the reigning sentiment. We proved one or two simple exercises involving congruent triangles. Then I gave an exercise to prove triangles congruent and no one gave me the correct answer. One by one they seemed to come to the conclusion that there was no need of hesitation in this class. The guesses became wilder. We tried the same exercise three days before someone, in disgust, said he didn't think it could be done. This was the case. The hypothesis was not complete. I didn't tell them *that* but made them figure out why it couldn't be solved. If they wanted an angle equal to a certain angle, they said they were equal; either assigning no reason or giving a reason which did not agree with the facts granted. These triangles *had* to be proved congruent and evidently the end justified *any* means. Another kind of exercise which caused a mental upheaval was such as this: Are the diagonals of any parallelogram equal? Prove the truth of the conclusion. They could prove black was white if told to do so, but if they were *asked* if black was white they would debate the question and finally decide one way or the other. They tried from then on to use the hypothesis intelligently.

Gradually new propositions were added to those of the congruent triangles and were solved in class. A number of the class remembered the simpler proofs but if a question were put in the middle of their recitation they acted as if I had lost their place. One day, I asked why a certain construction line was used. One little girl volunteered this information: "That comes next." I repeated the question thinking she had misunderstood. She answered "I don't know." "All right, then continue the proof." She continued, giving a proof that sounded perfect, though she did not see what the construction line had to do with it. This and similar experiences showed me that often by happy chance or careful memorizing, but not because they belonged there, statement and authority were put side by side. In giving reasons, no shortened forms were allowed and corrections in recitations were encouraged. "How does the authority fit the statement" was one of the eternal questions on my part. Often

they did not know. During one recitation of a given book proof consisting of two parts independent of one another, followed by the conclusion, I started the proof, for the pupil who had been called on, and suddenly was interrupted by, "Pardon me but if you don't mind I would rather give *that* part second so I won't get my authorities mixed."

Every day we reviewed either all the authorities we were at liberty to use, or all theorems from a certain book, or all theorems ending with certain words as "The chords are equal." Sometimes I wrote on the board eight to ten or even twenty theorems and had them arranged in order, not by memory but according to the nature of their proof. From this they saw how the theorems were linked. For rapid work in showing the relation between book proofs, the most effective way I found was by drawing the figures in air. In doing this I used to have them answer in general terms or I would point out certain things and ask what they knew about them. It is a great help because it holds attention, strengthens imagination and permits of very rapid review of proofs. To test their attention, after they became familiar with this method, I used to point to certain things and say "These are equal why?" Sometimes, they were caught napping when the wrong thing had been pointed out. I discouraged their judging by inflection of my voice the probable answer. I used to lead them on to wrong conclusions to diminish this excessive dependence.

One of the most difficult things I have ever faced was to help these people to solve originals not depending on congruent triangles. Unless I helped, they decided they couldn't do them. If I did help, they memorized what I said. Many could not keep the goal in mind or would write a proof that was true so far as the hypothesis was concerned but did not fit the conclusion. I suggested working on rough paper as follows: make two columns. Label first "What I know." Label the second "What I want to know." Under the first were put the granted and all the conclusions that naturally came from it. Under the second were put the statement to be proved and the possible ways of solving it, that is all theorems that end with statement to be proved—the second last steps so to speak. It was of course a rather lengthy process but a sure one and the connec-

tion between the second step and one of the second last steps was usually within their power. This constant review of related theorems kept them in use and gradually this working out in rough became mental. A certain number of originals were due at a certain date. All above this number received extra credit. The pupils came to talk over mistakes so they could see *why* they were *wrong* as well as what was *right*.

Certain days we reviewed in class a group of exercises which had been solved as originals. Credit was given to the person finishing the proof. They were somewhat familiar with the proofs but of course had not memorized them. They could interrupt after due warning if the pupil reciting made any mistake in idea. They often noticed things which were really unimportant as leaving out an authority where it is implied. The retaliation was very amusing and everyone liked to share in it. Often the credit came to someone who did not have the courage to go through an entire proof but who did not hesitate to come in on the last statement. This mark of credit used to encourage greater effort next time.

In marking proofs in tests, I marked on Regents' basis. The students used more care when they saw how much they lost by omitting important authorities or by quoting so inaccurately as to change the sense or by omitting part of granted, etc.

There was in this class a youth who had been taking geometry for five years. He could play the piano, play football and make a good speech. He did well in English but the boys used to rail him about his extreme fondness for geometry. At the beginning of the term he came to me and said he believed he had never really worked in geometry but he was determined to get the examination this time. "Now how do you do this original?" he asked me. I had warned the class to be definite in their questions to know what was granted and what to be proved and write the proof as far as they could. So I said "What do you know?" With restrained wrath he exclaimed "If I knew I wouldn't have to ask." His idea of help was for me to say the whole proof off. At first he would say a line equalled a line "because I can see they are equal." He used to answer at random. Sometimes I asked him how much he would risk on the truth of his statements. Gradually he raised his estimate

of the truth of his statements. My chief conversation with him used to consist of these questions "What do you know? What do you want to know? Have you used everything granted?"

The morning of the examination I saw him in the hall. "Well what did you think of the examination?" I asked. He replied, "I kept saying to myself what do I know? what do I want to know until I was afraid I would find myself shouting it aloud. Another thing, I would stake my life that I got it this time." He received the highest mark in this class, 91. I sent him a card with his mark and my signature and within the next two days I saw him several times show that card to someone. *They had to be shown.*

Eighty-two per cent. of the class passed the Regents' examination and seemed pleased to have mastered at last that very difficult subject.

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DECOMPOSITION INTO PARTIAL FRACTIONS BY MEANS OF REMAINDERS.

BY EUGENE RANDOLPH SMITH.

It has always seemed to the author of this paper that there should be some extension of the method of solving partial fractions by means of substitution, so that this method would be applicable to all cases without introducing any undue complexity. After several abortive attempts along the line of substitutions, the following method by the equivalent process of taking remainders was found.

There can be no claim that the method is entirely new, for other writers, for example, Chrystal, have used some of the same ideas, and have gotten some of the same results, though by somewhat different processes. The arrangement of the work in some of the cases is, as far as the author knows, an original one.

The proof for all cases is based upon the fact that two identical expressions, when divided by the same divisor, must give identical quotients and identical remainders. In certain questions this theorem is applied to several divisions taken successively.

The method is simple in application, presenting very little, even of arithmetical calculation, in ordinary fractions. The fractions having unfactorable (into rational factors) denominators present more algebraic difficulty than other types, but one of these of the n th degree requires only the solution of $(n - 1)$ first degree equations in the same number of unknowns.

The method can be applied to find the numerator for any one factor of the denominator (except powers of the same quantity) independently of the other factors.

The following illustrations will show the application of the method to the various types. In the division used in this paper the "multiply and add" method of synthetic division will be used for first degree binomial divisors, and successive remainders in the numerator will be represented by R_1, R_2 , etc., and in the denominator by r_1, r_2 , etc.

To decompose into partial fractions:

$$\frac{x^5 - x^4 - 25x^3 + 68x^2 - 13x - 38}{(x-1)(x-2)^3(x^2+x+2)}.$$

I. To determine the numerator for a denominator of the type $ax+b$: in this case, to find the numerator A for the denominator $(x-1)$.

Divide the numerator, and all of the denominator except $(x-1)$, by the required denominator $(x-1)$. Call the remainders R_1 and r_1 .

Numerator.	Denominator.
$\begin{array}{r} 1 - 1 - 25 + 68 - 13 - 38 \\ 1 \overline{) 1 \quad 0 - 25 + 43 + 30} - 8 = R_1 \end{array}$	$\begin{array}{r} 1 - 2 \quad 1 + 1 + 2 \\ 1 \overline{) 1 - 1 \quad 1 + 2 + 4} \end{array}$

$$\text{Therefore } (-1)^3 x(4) = -4 = r_1.$$

Then $r_1 A = R_1$, that is, $A = \frac{-8}{-4} = 2$, and the fraction is $\frac{2}{x-1}$.

In this type the method is evidently the same as the substitution method except that the results of substitution are obtained by division.

II. To determine the numerators for successive powers of $ax+b$: in this case the numerators A , B , and C for the denominators $(x-2)^3$, $(x-2)^2$, and $(x-2)$ respectively.

Multiply all the factors of the denominator of the given fraction except $(x-2)^3$, obtaining $x^3 + x - 2$.

Divide the numerator, and the product obtained in the denominator, by $(x-2)$, the quotient by $(x-2)$ again, and so on, until as many divisions have been performed (in this case three) as there are numerators to be obtained.

Numerator.	Denominator.
$\begin{array}{r} 1 - 1 - 25 + 68 - 13 - 38 \\ 2 \overline{) 1 + 1 - 23 + 22 + 31} + 24 = R_1 \\ 1 + 3 - 17 - 12 + 7 = R_2 \\ 1 + 5 - 7 - 26 = R_3 \end{array}$	$\begin{array}{r} 1 + 0 + 1 - 2 \\ 2 \overline{) 1 + 2 + 5} + 8 = r_1 \\ 1 + 4 + 13 = r_2 \\ 1 + 6 = r_3 \end{array}$

Then

$$r_1A = R_1, \text{ which gives } A = \frac{R_1}{r_1} = 3,$$

$$r_1B + r_2A = R_2, \text{ which gives } B = -4,$$

$$r_1C + r_2B + r_3A = R_3, \text{ which gives } C = 1,$$

and the fractions are

$$\frac{3}{(x-2)^3} - \frac{4}{(x-2)^2} + \frac{1}{x-2}.$$

As r_1, r_2 , etc., are coefficients of A, B , and C in order, these equations are easily remembered. There are several evident ways to write the six numbers with A, B , and C so as to find the results with a minimum of calculation. It is evident that each equation contains but one unknown, and that the first step of the method is identical to the method of I.

III. To determine the numerator for an unfactorable denominator of degree higher than the first: in this case the numerator $Ax + B$ for the denominator $(x^2 + x + 2)$.

Multiply together all the factors of the denominator except $(x^2 + x + 2)$, obtaining the product $x^4 - 7x^3 + 18x^2 - 20x + 8$.

Divide the numerator, and the product obtained in the denominator by $(x^2 + x + 2)$. After obtaining the first remainder in the denominator, multiply the dividend by x and divide again, obtaining another remainder. If there are three unknowns in the required numerator, the dividend must next be multiplied by x^2 , and again divided, and so on, until as many divisions have been performed in the denominator as there are unknowns in the required numerator.

In practice, these divisions can all be done in one, as the first division is identical in all of them, and the annexing of a zero detached coefficient raises the degree by one. The successive remainders are not first, second and third remainders as in the previous case, so will be indicated by r, r', r'' , etc.

	Numerator.	
	1 - 1 - 25 + 68 - 13 - 38	
- 2	- 2 + 4 + 50 - 194	
- 1	- 1 + 2 + 25 - 97	
	1 1 - 2 - 25 + 97 - 60 - 232	R

	Denominator.	Second Division.
	1 - 7 + 18 - 20 + 8 + 0	
- 2	- 2 + 16 - 48 + 56	
- 1	- 1 + 8 - 24 + 28	
1	1 - 8 + 24 - 28 - 40	r
		- 12 + 56 r'

Equating the independent terms, the first degree terms, etc., of the remainders by the rule, $rB + r'A = R$, sufficient equations to determine the unknowns in the required numerator will be found. (For three unknowns, the rule would be $rC + r'B + r''A = R$.) In this case, $-40B + 56A = -232$, and $-28B - 12A = -60$, giving $A = -2$, and $B = 3$, and the fraction is

$$\frac{-2x + 3}{x^2 + x + 2}.$$

Certain special cases, particularly where a denominator is a perfect power, merit attention.

(1) To decompose $\frac{8x^2 - 18x + 14}{(2x - 3)^3}.$

Divide three times by $(2x - 3)$, and the successive remainders are the required numerators. If $x - (3/2)$ is used as the divisor, remainders after the first must be divided by successive powers of 2 to equal the remainders for $(2x - 3)$.

$$\begin{array}{r|l} 8 - 18 + 14 & \\ 3/2 \quad 8 - 6 + 5 = A & \\ \quad 8 + 6 = 2B & \\ \quad \quad 8 = 4C & \end{array}$$

The numerators are 5, $6/2$ or 3, and $8/4$ or 2, giving

$$\frac{5}{(2x - 3)^3} + \frac{3}{(2x - 3)^2} + \frac{2}{(2x - 3)}.$$

(2) To decompose $\frac{3x^3 - 2x^2 + 10x}{(x^2 - x + 2)^2}.$ Divide by $(x^2 - x + 2)$

$$\begin{array}{r|rrrr}
 & 3 & -2 & +10 & +0 \\
 -2 & & & -6 & -2 \\
 \hline
 1 & 3 & & 1 & \\
 \hline
 1 & 3 & +1 & +5 & -2 = R_1.
 \end{array}$$

The first remainder is evidently $5x-2$, and the second remainder is the same as the quotient, or $3x+1$, giving

$$\frac{3x+1}{x^2-x+2} + \frac{5x-2}{(x^2-x+2)^2}.$$

(3) To decompose $\frac{6x^4 - 14x^3 + 29x^2 - 32x + 12}{(x^2 - x + 2)^2(x - 2)}.$

This requires a combination of methods II and III in finding the numerators for the powers of (x^2-x+2) . To find the numerator $Ax+B$ for (x^2-x+2) , and $Cx+D$ for $(x^2-x+2)^2$, divide twice by (x^2-x+2) .

Numerator.

$$\begin{array}{r|rrrr}
 & 6 & -14 & +29 & -32 & +12 \\
 -2 & & & -12 & +16 & -18 \\
 \hline
 1 & +6 & -8 & +9 & & \\
 \hline
 1 & 6 & -8 & +9 & -7 & -6 = R_1 \\
 -2 & & & -12 & & \\
 \hline
 1 & & & 6 & & \\
 \hline
 1 & 6 & -2 & -3 & = R_2
 \end{array}$$

Denominator.

$$\begin{array}{l}
 1 - 2 \quad r_1 \\
 0 \quad 0 \quad r_2
 \end{array}$$

Denominator multiplied by x .

$$\begin{array}{r|rr}
 & 1 & -2 & +0 \\
 -2 & & & -2 \\
 \hline
 1 & & & 1 \\
 \hline
 1 & 1 & -1 & -2 \quad r_1' \\
 & 1 & & r_2'
 \end{array}$$

Then, using r_1 with B and r_1' with A , $B-A=-7$ and $-2B-2A=-6$, and $A=5$, $B=-2$.

Also, using r_1 with D , r_1' with C , r_2 with B and r_2' with A , $A-2C-2D=-3$, $-C+D=-2$, giving, since $A=5$, $C=3$ and $D=1$.

(4) To decompose a fraction whose numerator is of higher degree than its denominator.

The method for any single partial fraction is exactly the same as in the cases shown. If the fraction is to be completely decomposed, the method will accomplish this automatically if the denominator is a perfect power; otherwise it is best to divide the numerator by the denominator as the first step.

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MATICS AS USUALLY GIVEN AT
THE PRESENT TIME?

BY MAURICE J. BABB.

The director of one of our most prominent laboratories of anatomical research told me during this last month that he has been having a frequently recurring and vivid dream for a year or more, to the effect always, that somehow he had failed to make good in mathematics, and that discovery was both imminent and inevitable. He confessed the sense of guilt and horror of exposure was so real as to be depressing even after he awoke. I have been wondering whether he is a sort of vicarious sufferer for the sins of his profession in particular and of this age in general. It is good to find some evidence of the arousal of a mathematical conscience. Simultaneously the much-maligned appendix vermiformis has been found to have unique and valuable functions, to the great consternation of numerous and fashionable people. All this is personally particularly interesting, as my first visit to the University of Pennsylvania was as a prospective medical student and naturally I drifted into the laboratory of the late Professor John Rider, whom you doubtless remember was rejected in his youth as unable to appreciate and absorb the knowledge as then dispensed by the adjacent public schools. He told me that he recently had been a student along with the sophomores in calculus in order to gain sufficient knowledge of the mathematics to be able to express the cellular development of the ear from the brain in embryo mice. This year a student of my calculus class comes in the same way that he may gain knowledge which he may apply to his researches on muscle fibers. These men were said not to have had mathematical minds in their school days. Professor David Eugene Smith has recently given the New England branch of this Society a list of seemingly hope-

less mathematical outcasts which on further examination has turned out to be a list of the great men of the science.

In early days one was a mathematician having been first a philosopher. And methinks, as I scan the pages of mathematical history, that this was true of practically all who have really extended the science.

When I turn to Poincare whose recent death deprived us of our greatest modern mathematician, I find, "The very possibility of the science of mathematics seems an insoluble contradiction. If this science is deductive only in appearance, whence does it derive that perfect rigor no one dreams of doubting? If on the contrary, all propositions it enunciates can be deduced one from another by the rules of formal logic, why is not mathematics reduced to an immense tautology?"*

For years, much more so since the Civil War, our elementary mathematics was taught by preachers who had strained their voices, and others, with here and there a thoughtful soul who appreciated the subject, knew something of real mathematical rigor, yet thought it no crime to "pass over rapidly, but without skipping stations, the road traversed slowly by the founders of the science." Poincare later asks, "Why is so long a preparation necessary to become habituated to this perfect rigor, which, it seems, should naturally impress itself on all good minds?" This is a logical and psychological problem well worthy of study, but we shall not take it up.

A centipede was happy quite
Until a toad in fun
Said, "Which leg moves after which?"
This troubled him to such a pitch
He fell exhausted in a ditch,
Not knowing how to run.

Suffice it to say that mathematics is a sequence subject in more ways than one and that there is a mathematical dialect. Blessed is the teacher who is student enough to appreciate more and more the beauties of its rigor and still remember the time when he would have deemed these same beauties irksome subtleties. Such a teacher will not forget that a real leader rides not too far ahead. All capable investigators are usually clearer of state-

* "Science and Hypothesis."

ment than their satellites and most of us will already agree that particular abilities are not necessary for the pupil to gain an understanding of the elementary and secondary mathematics as the average teacher understands them.

Until education became so general the comparatively few privileged pupils partook in all meekness the course prescribed for them. If there was not a place for the finished product, the community tolerated it as the Indians did those deemed similarly misfortunate. It is only when the mass moves up a stage that new adjustments are needed.

Probably in no science was there less concern about pedagogy than in mathematics. To-day we are asking, "Can literature be taught?" If the same question were asked about mathematics, what would be your answer? In my own experience as a teacher and private tutor I have yet to find an individual who failed to appreciate and enjoy mathematical reasoning when the simple underlying principles were made definitely clear. To do this needed both an understanding of the problem and its setting, both in the subject and in the mind of the pupil.

Most of our opinions to-day are founded on our limited personal observation. That personal observations are somewhat narrow is evinced by the following from over in Salisbury, N. H. The school committee was holding a meeting to consider the purchase of new text-books. An old man with a wobble in his chin and a tremble in his voice arose and said: "Didn't we study out of those books? Ain't what was good enough for us good enough for you? What's the good of book-learnin' anyway? We need something practical. I mind the time when I used to come along the old North road and see Dan Webster up in his gable window with a lamp aburnin' and him writin' and studyin' out o' his books as late as nine o'clock at night. What did it all amount to? Why he went away from here and we never heard of him afterwards."

Recently some investigators, notably Stone and Courtis and Thorndyke, with whose work you are familiar, have further simplified their test and endeavored to cover a wide field of observation. Our own arithmetic committee has meant to add considerably to our knowledge but will probably have to call upon well-equipped research laboratories and it will be some time before re-

liable deductions will be in the hands of the general teaching body. The handling of statistics is a subject on which I do not wish to speak at present. A very interesting recent investigation by E. O. Lewis attempts to find the most popular and the most unpopular subject by asking questions of 8,000 boys and girls in London and in South Wales. Some of the reasons are interesting; many are vague. But preference generally is clear-cut and the same opinions after as before the long vacation. The comment on grammar is: "I don't like grammar because I am not good at guessing." Such a remark is as pathetic as it is true. Scripture, which is the only subject to precede mathematics in the order of the day in England, was dismissed with: "Scripture will be no use to me after I am fourteen." One said: "Arithmetic prevents the shop-keeper from cheating us." Another said: "I don't like arithmetic, because there is no fun in it." In all the papers arithmetic was placed either very high or very low. So in the general average it occupies a position slightly above the middle. Here the average is not significant. An examination of the grouping seems to bear out what we have hinted before, namely, that where the underlying principles are well understood, mathematics is pleasing. But where these are not thoroughly clear the very discipline of the subject makes the pupil disgusted with the futility of his parody on reasoning.

It is said to be proverbial in America to tell what we intend to do in the next five years, whereas the German under the same conditions reviews what he has done during the last five years. Probably another paper may deduce from this fact why we do less mathematics in two and one tenth school years than the Germans do in one and two tenths. What causes the reversal of the digits? A popular American institute speaker once confessed to me his secret and gave me the following recipe. He said, "Choose a subject about which you know nothing and about which you are reasonably sure your audience knows nothing. Then you will not only have full freedom for any flight of fancy but you can easily flatter your hearers by your assumption of their familiarity with the subjects treated and with no danger of their exposing you, as they probably will be afraid to confer except in generalities." Many subjects in our curriculum may be placed in similar categories. But in mathematics the flow of

eloquence is subject to disconcerting questions and checks. In many cases the sociologist cannot be satisfactorily contradicted until long after his epitaph has disintegrated, for no one knows why the wildest dreams may not come true in this wonderful world of ours. The reasoning in mathematics is obvious and checks are possible. It is the simplest sort of reasoning known. Mathematics has always been a science,—crude at first, so as not to harrass beginners. As one reaches more maturity, of course the primitive mathematical notions will undergo elaboration so profound that it becomes difficult to analyze them. Yet there is a gradual growth of breadth of vision. In other subjects the reasoning is contingent, this and that have to be weighed and balanced. The skilled chess player may see four or five moves ahead. He wins if he can see more than his opponent. In chess the moves are limited. If the moves are unlimited how much better is one who sees four particular moves than one who sees three? The younger the subject, the less exact the data, the less probable is exact diagnosis. The guess of the tyro may ring as true to the mob as the brief of the expert.

Is it not probable that the objection to mathematics is that like the Spartan we object to being found out, to finding ourselves out and having this fact shamelessly confronting us?

"The elements of mathematics usually taught," says that modern man, Augustus de Morgan, "are contained in the sciences of arithmetic, algebra, geometry and trigonometry; we have used these four divisions because they are generally adopted, though in fact algebra and geometry are the only two of them which are really distinct." Let us examine these subjects taught in slight detail.

In arithmetic expertness in the four fundamental rules depends upon continued practice. Much of greatest common divisor and least common multiple is no longer needed, since the introduction of decimals. Fractions offer the first real difficulty to the individual as they did to the race. I went to school with a boy who although he is now a builder of repute has never ventured to get on familiar terms with fractions. A great philosopher was Mother Goose! Here a teacher who knows more about fractions than he will teach his class and in so doing understands his different pupils' peculiar points of view will

straighten out each mind, and extend the primitive ideas. The sequence of signs $+$ $-$ \times \div is pure definition. The decimal point can be corralled if trained early by a logical rule such as only computers use. If denominate numbers are not simple, with but a single reduction, it is probably a sign of particular ability if the pupil rebels. Carpeting, paper-hanging and a lot more so-called practical problems, are not worked as in the arithmetic in practice. They belong to special trades whose usage it may in some cases be desirable for the teacher to learn from local tradesmen and suggest such inquiries subsequently to his pupils as a practical field of mathematical research. Business arithmetic resembles very slightly the type found in books and had also better be replaced by local problems. A friend of mine who settles estates and transfers property for a large trust company tells me he always finds that people have the transactions figured out to within a few cents, everybody in his own way, of course, but all sufficiently exact to prevent being cheated to any great extent. They realize when dealing with their own money the absurdity of the question of the young clerk in the banking office when his employer took him to task for making a mistake of a thousand dollars against the firm in calculating some exchanges. The boy's answer was: "Wasn't the principle correct?"

In all this work much of the tendency to errors common in our department stores, phone exchanges and general clerical work is due to too much paper in the schools, especially scrap paper. It is probable if our schools would teach pupils to hand in their work as originally entered and generally in ink, from the first grade up, that the next generation would show more improvement than many previous periods. If added to this the German method of chalk and talk were used, mathematical shorthand would take its proper place and oral English would be greatly improved. We used to have mental arithmetic till it became unreal. Then we figured and figured and wrote out long statement forms. What we want now is team work with chalk and talk. With this, transition to algebra will come easier. Written mathematics should be an obviously needed and suggestive shorthand for problems not easily rendered orally. Particular ability of the pupil is not needed here but better judg-

ment on the part of the teacher and text-book maker. Even with an inferior text-book the chief trouble is in judgment on what to retain. Logarithms are about the only really excellent thing subjectively excluded from the arithmetic course. With proper training in arithmetic, algebra is but a more complete and generalized system of shorthand,— 563 is $5(10)^2 + 6(10) + 3$ and not particularly different in conception from $5x^2 + 6x + 3$. Here again if the teacher is familiar with the history of the race's difficulties he can guide his pupils past blind alleys. Always he should remember that it is desirable for him to know much more than his pupils, and also it is even desirable in many cases that his pupils should not be taught as yet as much as he knows about particular subjects. Many teachers run to seed with too many kinds of factoring and then omit the factor theorem altogether. Let us here, as in arithmetic, also avoid the errors that come from mental tire caused by too long continued concentration. We should choose a few well-selected methods, always keeping in mind the significance of the result rather than the trick of guessing it. For example, very few pupils are taught what the root of an equation is. When principles are clear, complications may be introduced, but efficiency in manipulation of symbols is always a matter of practice. It cannot be retained inert, but seems easily regained. In radicals and imaginary numbers again the teacher must assure himself that he understands what he is doing and convince himself that he is not trying to transfer a habit confirmed by years to youthful minds in so many weeks. It is well to understand the principle of permanence, or of no exception.* Completing the square can be more mechanical than the formula. If proper drill has been had on roots the theory of quadratics will be enlightening and will make an impression for future use in analytical geometry. Simultaneous quadratics need geometric interpretation to show how special a case they are, depending entirely on recognition of forms. Arithmetic needs co-ordination here and continual review can be had in verifications and substitutions in formulæ. Operations used in arithmetic should be recognized, generalized and extended. Here again the teacher should not forget that he ought to know more than his pupils

* Schubert, "Mathematical Recreations and Essays."

and not be proud of it. Why shouldn't he? Somewhere in here comes the geometry, making use of much of the algebra and supplying problems for the algebra, thus early showing that there are obverse and reverse sides of mathematics.

Strange to say, geometry often seems to give more trouble to really good inquiring minds than to mediocre memorizers and dull acquiescent brains. The tacit hypotheses distributed all through the geometries we use do not agree with the pupil's particular manner of reasoning and he is confused and sometimes righteously belligerent. The English student attacks his ancient enemy Euclid single-handed, but with this hand in that of his tutor who knows his quality of selected mind. Here, too, after the geometric habit is formed, more theorems should become exercises and special cases generalized. All through mathematics too many things are committed to memory which would logically follow without effort. The pencil is used too little on the set proofs and too much on the problems.

Trigonometry does not consist merely in solving triangles. Indeed this is a minor part. The symbols of arithmetic and algebra are again combined with an added set of operators.

Let us then have less memory, more team work of chalk and talk and more cultivation of imagination and observation in our mathematics

"No," complained the Scotch professor to his students, "ye dinna use yer faculties of observation. For instance," picking up a jar of chemicals of vile odor, he stuck first a finger into it and then put his finger into his mouth, "taste it yourselves," he commanded. After each had licked his fingers and had felt rebellion through his whole soul, the professor exclaimed triumphantly, "I told ye so. Ye dinna use your faculties, for if ye had observed ye woud ha seen that the finger I stuck into the jar was nae the finger I stuck in my mouth."

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MATHEMATICS: THE SUBJECT AND THE TEACHER.*

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These are days of change in educational affairs. Greek has almost disappeared from the curricula of school and college. It has entirely disappeared from the individual schedules of all save a mere handful of our students. The contest for the survival of Latin is being hard fought and some of its champions prophesy early failure. Very few men's colleges in New England any longer demand for admission to the Arts course the traditional four years of Latin. The Bachelor of Science course without any Latin at all has become the common thing. We in the far east will hardly be allowed by ourselves to decide the question of the survival of Latin; the great state universities of the middle west promise to play the chief part in the solution of such educational problems, and we probably shall feel obliged to accept their verdicts. Verily "the old order changeth, yielding place to new." Is one to see all the old-fashioned subjects drop out in their turn, and will mathematics soon depart forever from the student's schedule? It requires no excess of confidence to answer no to this question. One notices that the onslaught of the progressives with their weapons of introductory social science, introductory general science, practical arts, community civics, typewriting, forging, and domestic science is aimed at the foreign languages, particularly Latin, and not at mathematics. A careful examination of several of those new curricula which promise to make the high school course really useful, and at the same time easy and delightful to the pupil, shows that they retain in every instance as required subjects for all students algebra and geometry. Naturally these two branches of mathematics cannot be expected to add to the ease or largely to the delight of the average boy or girl's high school course; they are distasteful

* A paper read at the winter meeting of the Syracuse Section of the Association of Mathematical Teachers of the Middle States and Maryland at Syracuse, December 30, 1913.

to many an earnest but non-mathematical student; they have no charm of educational novelty; yet there they stand as subjects required of all secondary school boys and girls, and why? Because they are known to be in many small ways useful subjects and in a large way necessary subjects. As to the future, one may feel sure that mathematics, which began with the first ordering of things, will not pass out from its place in our educational life until such time as order in the universe shall have given way to chaos. For mathematics, as Professor Story says, occupies a peculiar position with respect to both the arts and the sciences. It is, *par excellence*, an art, inasmuch as its chief function is to solve problems—not textbook examples which are merely exercises in the application of methods—but all those problems in human experience for whose correct solution sufficient data are at hand. When any line of investigation, to whatever subject it may refer, has been carried so far as to permit the application of exact reasoning, mathematics assumes control. It determines whether the results presented are consistent and whether the conclusions are valid, and it is mathematics that provides the means for applying these conclusions to the prediction of phenomena not yet observed. No science and no branch of technology is exact unless it rests on a mathematical foundation. The secrets of the future can be foretold for any science only in so far as that science is obedient to the laws of mathematics. Astronomy long ago had accumulated enough data of observation to allow its case to be submitted to the mathematician and accordingly astronomy began long ago to be an exact science. It was the mathematicians, Adams and Leverrier, who told the astronomers that their data called for another planet, and indicated to them the direction in which the astronomer should point his telescope to find it. After astronomy, next came physics, and then applied mechanics, in both of which a sound mathematical foundation has long been in building. Other sciences are in the inductive stage, still collecting material to be put in order by the mathematician as soon as the material is sufficient for the purpose. The calculus is already being applied to some of the problems of economics. The chemist employs mathematical formulas daily. The geologist, the physiologist, and the psychologist all use mathematical tools. And mathematics must continue to be used for

such tasks until all science shall have become perfect and shall have passed away forever. Mathematics, then, is one of those few themes which, in Kipling's phrase,

" Shall be sung through planets young
When this is clean destroyed."

But mathematics is not only an art, weighing and arranging and interpreting the data of the sciences. It is also a science, since it has accumulated a vast store of systematic knowledge involving and leading to the methods that it uses in its solutions. These methods are of a very peculiar nature. They differ so widely from all other methods that a special course of training is necessary to him who would use them. And these methods are so many and so varied that no scholar lives long enough to become familiar with them all. Furthermore, new problems are continually arising of such sort as demand new methods; hence mathematics needs the continued service of able minds. Even if the solution of the quadratic equation and all involved therein has been completely determined, the same is probably untrue of the equations of degree three and surely untrue in the case of equations of all higher degrees. Since one may add new variables in countless number to the equation of any degree and may extend the degree in each instance indefinitely, the labor still awaiting the mathematician surpasses the reach of the imagination. It will hardly have been begun when Kipling's "planets young" shall in their turn be "clean destroyed." Mathematics is, then, an art and a science; and its task in either capacity and in both will endure from "everlasting to everlasting."

This estimate of the subject of mathematics is not the one commonly held by the uninitiated. The popular view of mathematics still associates something queer with its name and with those who are its devotees. No doubt it is true that unsound dabblers in the use of mathematical symbols have contributed more than their share to that little collection of books classified in a university library as "Insane Literature." It is there that you will find the "Life Romance of the Algebraist," a remarkable work, which mingles mathematical symbols with sentimental phrases and attempts to set the combination to music; its mathematics is unsound, its verses are unpoetical, and its music

is unharmonious. The author recites in the strange medley the hopelessness of his attempts to accomplish two objects,—namely, to win a young lady's affection, and to solve some difficult equations; a rival forces him to relinquish his first object, but no one robs him of his equations, and he continues his vain struggle with them to the end of the crazy volume.

But mathematics has to bear the devotion of the ignorant even more persistently than that of the unbalanced. The problem which best illustrates this is the famous one of "squaring the circle." This demands a method for the construction of a square which shall be exactly equivalent in length of perimeter or area to the circumference or area of a given circle. Thus it involves the ratio of the circumference of a circle to its diameter. It is an ancient problem. One reads in the seventh chapter of the first book of Kings that Hiram of Tyre found this ratio in the case of a molten sea "round all about" to be as "three to one." Aristophanes in the "Birds" introduces a geometer who announces his intention to "square the circle," while Pope, in the "Dunciad," declares the person "mad" who attempts a solution. Archimedes, by a method declared "more divine than human," obtained a close approximation to the 3.14159 which we use. Ptolemy carried the approximation of the famous π still further, while Van Ceulen gave much of his life to its solution and directed just before his death in 1610 that his best result of 35 places of decimals be engraved on his tombstone in St. Peter's Church, Leyden. In our own era Mr. Shanks has extended the result to 707 places,—a line of figures nearly six feet long when set in ordinary type,—and constituting an approximation sufficiently close for any purpose. About twenty-five years ago a German mathematician, Lindemann, established forever that the value of this ratio cannot be given in exact terms. No two numbers whatsoever can represent exactly the ratio of the circumference to the diameter of the circle. Accordingly the circle cannot be squared. But this demonstration has not caused much diminution in the number of so-called solutions printed and distributed throughout the world. Hiram of Tyre was not the first to wrestle with this problem, and the world has no prospect of seeing the last. For some centuries it was supposed that a great reward had been offered by the Queen of England, the French

Academy, or some other distinguished organization for the first correct solution. The great, as well as the ignorant, labored long over it. Some by prayer and supplication sought to reach the desired goal; many a priest in the middle ages reported that he found the problem most highly conducive to his spiritual welfare, and the French mathematician, DeVausenville, announced in 1771 a solution obtained by the direct aid of the Holy Spirit. Several years after Lindemann had proved the problem insoluble, a writer in the *New York Tribune* announced that he had discovered the long-lost secret, the "Nicomedean line," and had established the desired ratio as exactly 32 to 10. It is reported that the legislature of a western state undertook to assist the working mathematician and dispose of the problem forever by fixing the ratio by legislative enactment at 3 to 1. Whether the circumference of the circle was to be forever contracted or the diameter was to be permanently stretched; whether all geometric laws were to be restated and all structures built on mathematical principles were to be shaken from their foundations by the passing of a majority vote in the capitol of a western state, one cannot tell, for the legislature finally decided to leave geometry as it is until, perhaps, the combined attack of several legislatures shake Euclid from his grave and force him to rewrite his famous books. The search for the "Fountain of Youth,"—at least among the native springs of the earth,—has become unfashionable; the chemists and geologists have made us despair of the discovery of the touchstone that shall give us transmutation of metals; electricity is accepted as the nearest approach to the long-sought "perpetual motion"; but the squaring of the circle will still command its earnest laborers. The pseudo-mathematicians will continue to burn their midnight oil over this question and will send through the mail their illogical results. Still the newspapers will publish the solutions of mathematical problems long since proved insoluble. Learning can annihilate learning, but learning cannot annihilate ignorance. "A sword may cut through an iron bar, and the severed ends will not reunite; let it go through the air, and the yielding substance is whole again in a moment."

Attention has been called to the fact that mathematics receives an undue amount of devotion, first, from the unbalanced, and, second, from the ignorant. From both these sources it is en-

veloped in an atmosphere of which it would perhaps gladly be rid. In the third place, the successful mathematician in many an instance allows the peculiarities of his own personal conduct to color the subject in the eyes of his associates. In particular, the professor of mathematics comes to be regarded as a most absent-minded fellow, whose thoughts linger in vast spaces of his beloved geometry when needed to direct his present movements. So it happens that the professor of mathematics lets his watch get two hours slow and forgets that it is not on time; he leaves a notice in his office window reading, "Will return at two o'clock," and hesitates to enter when he comes back at one-thirty but sits on the steps until he can be sure of arriving at the appointed moment. He forgets that he has declined an invitation to dinner at a friend's house because of another engagement which did not exist and wanders in aimlessness and complete forgetfulness to that very house on the evening of the dinner and lingers in street clothes in the drawing room until the arrival of the well-dressed diners reminds him to run away. It is a professor of mathematics who finally yields to his wife's entreaties to come home and accompany her on an evening's programme of social events; she rejoices at nightfall to see him mindful of his promise and tells him to hurry upstairs and dress so that they may be off; after patiently waiting long in vain for his reappearance, she seeks him in his chamber and finds that he has undressed and gone soundly to sleep in his bed. Numberless are these stories of the professor of mathematics and the most unpleasant thing about them to the mathematician is that they are all true. Probably the mind trained to deal directly with large questions in the field of abstract reasoning is quickest to lose its temporary grasp of the present and the visible; the journeying often in the great spaces of analytical geometry tends to make one absent-minded. Still waters run deep, and the mind engaged in tracing out the singularities of twisted curves in n -dimensional realms may fail to be stirred by the sound of the dinner bell. But one need not hesitate to enter the field of the mathematician's toil because of such facts as these, for one has suffered little from the effects of his profession when these are its worst results.

It would be a more serious reflection on mathematics if all of those unkindly criticisms which some of the mighty are fond of

making with regard to the teachers of that subject were deserved. A university president used to like to refer to those "least human of teachers, the mathematicians." It is granted that few teachers of mathematics are human to the extent of marking examination papers without reading them, or even of accepting the inaccurate statement as, by good intention, of the same value as the accurate one. The mathematician finds his subject an argument for precision and he aims at justice with full hope of reaching his goal. Let him not be blamed that he is keen at detecting errors and quick at calling mistakes by their proper names. In such direction has his subject trained him. Again, a university president stated to a little company of college officers that the teacher of mathematics was incapable of forming *moral* judgments. One of his hearers asked him how he would reconcile this statement with the fact that the deans of our colleges and universities are more largely taken from the professors of mathematics than from any other class of college teachers, and that these deans handle the questions of discipline which call constantly for the forming of moral judgments. He replied that he was not aware that the mathematicians were providing more than their share of these officers. In a truth-seeking spirit he at once investigated the matter, found that he was in error, and generously acknowledged the truth to the mathematicians among his associates. That a teacher whose subject demands clear reasoning and insistence on the reaching of definite, irrevocable conclusions should be found capable of dealing with cases of infraction of college law need surprise no one. The habit of reaching definite results and holding them fast is a good habit and such an attitude carries conviction with it. A boy gains confidence in the justice of his punishment if he finds that his sentence is held by the disciplinarian,—not angrily,—but ever so clearly as a deserved thing and ever so firmly as a necessary thing. For the development of this attitude, no better training is found than that afforded by mathematics.

But the text of this rambling discourse on mathematics includes the teacher as well as the subject. He is a strange and unventuresome critic in these days who cannot insist that the teaching of our schools is uninteresting, uninspiring, inefficient, unpedagogical,—a waste of money,—and that the teaching of

our colleges is still worse. With a confidence often born of inexperience, the reformer sets forth the faults in each layer of the whole structure of our educational system. What is easier to establish than the proposition that the high school teacher fails to make brilliant scholars of all the unwilling, happy-go-lucky youths who throng the classroom? or that the college teacher cannot make either a scholar or an athlete of many a youth who, for avowed purposes social, or athletic, but not educational, is sent by fond parents for four years of indulgent life in the college atmosphere? That the results are not nearly so bad as they are painted, all who teach are well aware. That, in view of the commercial spirit of the time and place, the lack of intellectual stimulus in the home, and the failure on the part of the public to appreciate what of good is done, the teachers are doing surprisingly well is a statement void of exaggeration.

Mathematics fares no better and no worse, perhaps, at the hands of critics than the other school subjects. Of it, one reads that it "has outlived its usefulness as a subject of secondary school instruction." "It is too remote from life to interest the student." "There is no such thing as mental discipline, hence mathematical teaching has no value." Such sentences in almost countless number are thickly scattered about in the journals of these days. Still, as Kant said: "A science is exact only in so far as it employs mathematics." "Our entire present civilization," says another, "as far as it depends upon the intellectual penetration and utilization of nature, has its real foundation in the mathematical sciences." Engineering, architecture, navigation, railroad building, and surveying all rest on a mathematical foundation. "There is no royal road to geometry," said Euclid; and when will our boys equip themselves with the necessary mathematics for their parts in our American life unless their teachers give them a good start in this great subject in their early days? Moving pictures are more entertaining for the moment than equations, but to entertain is not a teacher's chief function, the critics to the contrary notwithstanding. Not "what does Peter, the boy, wish now to study?" but "what will Peter, the man, wish twenty or thirty years hence that he had studied?" is the vital question. If he or she is never to come in contact with any development of science which will

demand mathematical insight, it still is true that the secondary school curriculum has no other subject nearly so well adapted to the exercise of the reasoning power as geometry. Other subjects may help to a larger acquisition of facts, but the study of mathematics excels in the development of power.

Those who teach freshman mathematics in college feel that the student is too insecurely grounded in the simpler and more commonly used principles, particularly in algebra. As a freshman recently explained, he had never been well grounded in the "fundamentals" of the subject. Truly, if all these boys had learned and could not forget the "fundamentals," the college teacher of mathematics would have a far happier career. The average freshman knows that he can solve for x the equation $2x=1$ and immediately, with unwavering confidence, he gives the result $x=2$. Similarly, the equation $x/2=1$ establishes to his full satisfaction that $x=\frac{1}{2}$. The chief need of the student for success in college algebra is the ability to solve quadratics; yet very few of them are at all trustworthy for the performance of that feat unless the equation is easily factorable at sight. One wonders whether it is not possible in school to drill these boys so long and so thoroughly in the solving of equations of the first and second degrees that they cannot forget, even in a long summer vacation. Yet when the college teacher meets the same boys in class in sophomore year and finds that his own systematic drilling of six months ago has gone to "that bourne whence no pilgrim returns" he thinks far less unsympathetically of the high school teacher. To repeat the instruction on the fundamental things again and again and yet again must be the mathematics teacher's task.

The department of mathematics at Williams has striven earnestly to improve its methods of teaching and has experimented with small changes in method from time to time. Some of these slight departures from the ways of teaching which were found there twenty years ago have now been thoroughly tried and permanently adopted. The student has need of much practice in the application of principles and theorems; to secure this practice he is required first to present in writing at the beginning of every recitation, as an imperative part of that day's exercise, from six to twelve appropriate examples, chosen ordinarily from the textbook, and assigned at the previous recitation. The prob-

lems are carefully examined by the instructor and an accurate report of the results is posted on the bulletin board before the next exercise. Unless the student presents during the semester at least seventy-five per cent. of these assigned problems correctly solved, he is declared "deficient" at the end of the term and must repeat the course. This requirement is carried out to the letter and the student follows the posted record with great interest. To prevent the postponement of this part of the work and the consequent lagging of the student in the course it is required that at least half the assigned problems be presented in correct form by every student the very day when they are due. Furthermore, the few stray problems in which the student has failed may not be made up at all later than one week after they were assigned. The papers of problems thus submitted are not returned to the student, for it is counted undesirable to have sets of solved problems left permanently where a succeeding class might find them. The different divisions of the class have different assignments,—an arrangement which discourages copying. This laboratory feature of the work has been regularly employed in all the courses for at least a half dozen years and would not willingly be relinquished by any of the five teachers of mathematics. On the pupil's side, it has not at any time aroused much opposition, for the student finds that it is not without value to him.

Again, the department makes sure that no division should at any time include more students than can be accommodated, all together at once, at the blackboards in the recitation rooms which the department uses. Accordingly, a division seldom includes more than fifteen students, and the average number at the present time is less than thirteen. A recitation hour seldom passes without the instructor's sending the entire class to the blackboard to demonstrate theorems and solve problems there. To make sure that the students shall work independently, they are not allowed to take their books to the blackboard and are not permitted to confer with each other there, but are directed to call on the instructor for all needed help. Furthermore, each student has his own individual assignment at the blackboard. To make this possible and to secure that each student has a full and correct statement of his problems before him, the instructors prepare sets of cards for successive recitations. Each card bears

problems suited to the lesson of the day; no two cards are alike and the problems for each exercise are either made up by the instructor or selected from other textbooks than that used by the class. Naturally the problems are as varied and call for as much original ability as the instructor can safely demand. To distribute these cards to the class consumes but a moment of time, and each student finds himself at the blackboard face to face with a problem which he alone is to solve. The instructor has the answers on another card in his hand and goes from student to student checking the work, distributing fresh cards, pointing out errors, bestowing help,—actually teaching each individual boy as that particular one shows need. All students take kindly and earnestly to this method. Practically never is one of the cards lost and the sets in use now have remained unbroken in many instances through a half dozen years. None of the staff of teachers regards any other feature of the work of so great profit as this,—with the sets of cards, the large blackboards, and the small divisions.

Each student is called on for oral recitation nearly every day. With these three means of contact, the problems done in the study and presented on paper, the work done on the blackboard, and the oral recitations, the instructor knows what progress his students are making, what principles need further emphasis, and whether his rate of progress is adapted to his class. Teaching keeps its charm under such circumstances. It can hardly do so when the classes are far too large and the possibility of instructing the individual according to his needs is lost.

In some universities, the teaching of all the classes in mathematics is done by lectures. At Williams, the work of nine semester courses rests solidly on textbook foundations, while only a single senior course is taught by lectures. At the same time, in all the courses the more difficult theorems of the next lesson are explained somewhat in advance. This saves the student's time, for it permits him to undertake the study of the theorem in the textbook with some knowledge of the reasoning employed and the goal sought.

From various sources comes to every teacher the demand that his subject be made interesting and show contact with real life. How far can the teacher of mathematics legitimately satisfy this demand? In each branch of mathematics some ingenuity may

well be expended in selecting such problems as men actually meet in every-day affairs rather than those artificial ones which never have existed outside an old-fashioned textbook. The field of "industrial mathematics" so-called offers many suggestions in this direction for the use of the classes in algebra and trigonometry. In geometry, one may rouse much interest by leading a student to the consideration of spaces of zero, one, two, three, and even four and higher dimensions. One point determines a point in one-dimensional space; two points determine a line in two-dimensional space; three points determine a plane in three-dimensional space; what, then, by analogy, must four points determine in a four-dimensional space and five points in a five-dimensional space? The four bounding lines of a square, a two-dimensional figure, may be folded down into a one-dimensional space; so may the six bounding planes of the cube be folded down into two-dimensional space; what results if the eight bounding cubes of the corresponding 4-dimensional figure are folded down into 3-dimensional space? Time, as a one-dimensional space, always interests a class of freshmen. An inhabited two-dimensional space, as described in "Flatland, by a Square," with the resultant "Theory of Ghosts," when presented to a college class, proves a lively subject of discussion at many a student dining-table. Mathematics is not lacking in material to interest the student.

After all, the great lesson to be gained by the average student from the study of mathematics is accuracy. That the slightest departure from the straight and narrow path makes the result inaccurate and therefore wrong,—not almost right, but everlastingly wrong,—this is the great lesson for our students. In nearly all other subjects the "nearly right" may answer. The student is accordingly accustomed to a state of well-intentioned uncertainty, which impedes his progress and weakens his character. The teacher of mathematics must impress his pupil's mind indelibly with the fact that the exact alone is acceptable, while the inexact is a thousand times not acceptable. It is a good lot in life to be called to teach the young this lesson. And there is much satisfaction in dealing day by day with a subject where wrong is wrong and right is right forevermore.

WILLIAMS COLLEGE,
WILLIAMSTOWN, MASS.

NEW BOOKS.

Constructive Text-Book of Practical Mathematics. By H. W. MARSH.
Volume IV. Technical Trigonometry. New York: John Wiley & Sons.
Pp. 232. \$1.50.

This book is quite different from the conventional text on the subject and is designed for those who are to enter industrial work. The author tries to use the student's interests and no doubt those who work it through will know far more real trigonometry than the average student in the subject. Besides trigonometry he will know many things connected with machines and manufacturing which the average student will know nothing about. It is the result of several years' trial in the author's class-room.

A School Course in Geometry. By W. J. DOBBS. London and New York: Longsmans, Green and Co. Pp. 427. \$1.00.

This volume is the result of many years of thought and experience by an author who is well fitted for the work. It includes not only elementary geometry, but trigonometry and mensuration, and an introduction to coördinate geometry, including the elements and use of the calculus. Instead of proceeding from the congruence of triangles to establish fundamental geometrical truths the author uses the notions of rotation, translation and folding for which advantages are claimed.

The Algebra of Logic. By LOUIS COUTERAT. Translated by L. G. ROBISON. Chicago: The Open Court Publishing Co. Pp. 98. \$1.50.

This is much the simplest and most concise handling of a subject treated much more extensively and ponderously by other representatives of symbolic logic. It is well fitted to serve as an introduction to the study of mathematical logic. A preface by Philip E. B. Jourdain is of an historical nature and gives a very thorough account of the development of the subject and the various phases of it especially emphasized by the different representatives.

Memorabilia Mathematica or The Philomath's Quotation-Book. By ROBERT E. MORITZ. New York: The Macmillan Company. Pp. 410. \$3.00.

The author of this book has done a good work in making such a large and interesting collection of quotations concerning mathematics. They should be interesting not only to teachers and others working in the subject but to those whose interest is but indirect or remote. Anyone would have a keener and better appreciation of the subject after reading this book. It is arranged under the following heads: Definitions and Objects of Mathematics, Nature of Mathematics, Estimates of Mathe-

matics, The Value of Mathematics, The Teaching of Mathematics, Study and Research in Mathematics, Modern Mathematics, The Mathematician, Persons and Anecdotes, Mathematics as a Fine Art, Mathematics as a Language, Mathematics and Logic, Mathematics and Philosophy, Mathematics and Science, Arithmetic, Algebra, Geometry, The Calculus and Allied Topics, Fundamental Concepts of Time and Space, Paradoxes and Curiosities.

Elementary Algebra. Revised. By FREDERICK H. SOMERVILLE. New York: American Book Company. Pp. 447.

This book is a revision of the earlier Somerville Algebra, with some improvements and the addition of many new exercises. Among this new material are sets of recent college entrance examination questions. The development seems quite simple although the introduction of the problem is later than in many of the new books.

The preface calls attention to the "thorough treatment of the graph" and the book does give a simple and fairly comprehensive view of the simpler cases. It seems, however, as if a false idea of the general equations might be left with its pupil, as the text gives special forms as types for the various second degree curves; for example, $x^2 + y^2 = c$ is given for the circle.

One of the best features of the book is its lists of exercises, both with the topics and in review lists.

NOTES AND NEWS.

A NEW patented indicator or runner for slide rules, called the "Frameless" has just been perfected by Keuffel & Esser Co. Every figure on the rule is clearly visible at all times, there being no side pieces to the metal holder of the glass Indicator, and therefore, nothing to hide any of the figures on the rule (see fig. 1). This is one of the most important improvements in slide rules—those indispensable instruments for rapid calculations.

Many times, after setting the old style indicator or runner, the user would find that he could not read the result because important figures were hidden by the frame or holder of the glass (see Fig. 2). Fre-

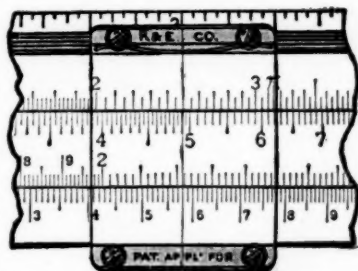


FIG. 1. *New Style Indicator.* Never Obstructs any Figures.

quently two, and sometimes, four or even six, eight or nine figures would be thus hidden; causing more or less inconvenience and uncertainty in reading the slide rule. The new K & E "Frameless" indicator or runner entirely obviates this difficulty.

This significant improvement vastly increases the ease and rapidity of using the rule. It will be keenly appreciated and enthusiastically welcomed by the thousands of users of K & E Slide Rules.

Hereafter, all K & E *Adjustable* Slide Rules of the Mannheim and Duplex type, will be regularly equipped with "Frameless" Indicators.

Thousands of engineers and scientists in every profession and industry, as well as contractors, builders, architects and merchants have found the slide rule invaluable as a time and brain saver in quickly and accurately making a great variety of calculations. In its various forms, it has been adapted by Keuffel &

Esser Co. to the needs of practically every line of work, including all branches of engineering, as well as chemistry.

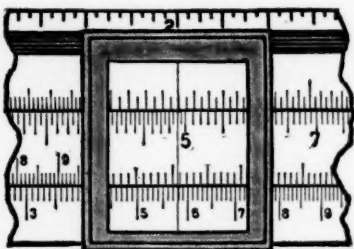


FIG. 2. *Old Style Indicator.* Hides Important Figures.

K & E Slide Rules are made of the finest obtainable material; an entire department of the large Keuffel & Esser Co. factories is devoted exclusively to their manufacture. Numerous features beside the new "Frameless" Indicator are patented and are not found in other rules. Such, for example, is the K & E Patented Arrange-

ment which permits of adjusting the rule to any friction desired between the slide and the rule; this successfully counteracts any tightening or loosening of the slide from atmospheric changes.

THE Council of the Association of Mathematical Teachers in New England has recently appointed a special committee on the status and welfare of mathematics in secondary schools, to investigate and report on current criticisms of high school mathematics. A fuller statement in regard to the personnel and plans of the committee will be made later. Correspondence with persons having special data on the subject is invited by the chairman, Professor H. W. Tyler, Massachusetts Institute of Technology, Boston.

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THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

Volume VI June, 1914 Number 4

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Published Quarterly by the

**ASSOCIATION OF TEACHERS OF MATHEMATICS
FOR THE MIDDLE STATES AND MARYLAND**

LANCASTER, PA., SYRACUSE, N. Y.

NEW YORK, N. Y., PHILADELPHIA, PA.
BOSTON, MASS. BALTIMORE, MD.

Entered as second-class matter October 7, 1909, at the Post Office at Lancaster, Pa.,
under the Act of March 3, 1879.

THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

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THE MATHEMATICS TEACHER is published quarterly—September, December, March and June—under the auspices of the Association of Teachers of Mathematics for the Middle States and Maryland.

The annual subscription price is \$1.00; single copies, 35 cents.

Remittances should be sent to the Syracuse office by draft on New York, Express Order or Money Order, payable to The Mathematics Teacher.

THE MATHEMATICS TEACHER,
41 NORTH QUEEN STREET, LANCASTER, PA.
OF SYRACUSE, N. Y.

